

# Projective Oscillator Representations of $sl(n+1)$ and $sp(2m+2)$ <sup>1</sup>

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## Abstract

The  $n$ -dimensional projective group gives rise to a one-parameter family of inhomogeneous first-order differential operator representations of  $sl(n+1)$ . By partially swapping differential operators and multiplication operators, we obtain more general differential operator representations of  $sl(n+1)$ . Letting these differential operators act on the corresponding polynomial algebra and the space of exponential-polynomial functions, we construct new multi-parameter families of explicit infinite-dimensional irreducible representations for  $sl(n+1)$  and  $sp(2m+2)$  when  $n = 2m+1$ . Our results can be viewed as extensions of Howe's oscillator construction of infinite-dimensional multiplicity-free irreducible representations for  $sl(n)$ .

**Keywords:** special linear Lie algebra; symplectic Lie algebra; oscillator representation; irreducible module; polynomial algebra; exponential-polynomial function.

## 1 Introduction

A module of a finite-dimensional simple Lie algebra is called a *weight module* if it is a direct sum of its weight subspaces. A module of a finite-dimensional simple Lie algebra is called *cuspidal* if it is not induced from its proper parabolic subalgebras. Infinite-dimensional irreducible weight modules of finite-dimensional simple Lie algebras with finite-dimensional weight subspaces have been intensively studied by the authors in [1, 2, 3, 4, 5, 6, 8, 12]. In particular, Fernando [6] proved that such modules must be cuspidal or parabolically induced. Moreover, such cuspidal modules exist only for special linear Lie algebras and symplectic Lie algebras. A similar result was independently obtained by Futorny [8]. Mathieu [12] proved that such cuspidal modules are irreducible components in the tensor modules of their multiplicity-free modules with finite-dimensional modules. Although the structures of irreducible weight modules of finite-dimensional simple Lie algebras with finite-dimensional weight subspaces were essentially determined by Fernando's result in [6] and Mathieu's result in [12], explicit structures of such modules are not that known. It is important to find explicit natural realizations of them.

Let  $\mathbb{F}$  be a field with characteristic 0 (say,  $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ ) and let  $n \geq 2$  be an integer. A projective transformation on  $\mathbb{F}^n$  is given by

$$u \mapsto \frac{Au + \vec{b}}{\vec{c}^t u + d} \quad \text{for } u \in \mathbb{F}^n, \quad (1.1)$$

where all the vectors in  $\mathbb{F}^n$  are in column form and

$$\begin{pmatrix} A & \vec{b} \\ \vec{c}^t & d \end{pmatrix} \in GL(n). \quad (1.2)$$

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It is well-known that a transformation of mapping straight lines to lines must be a projective transformation. The above transformations give rise to an inhomogeneous representation of the Lie algebra  $sl(n+1, \mathbb{F})$  on the polynomial functions of the projective space. Using Shen's mixed product for Witt algebras in [13] and the above representation, Zhao and the author [14] constructed a new functor from  $gl(n, \mathbb{F})\text{-Mod}$  to  $sl(n+1)\text{-Mod}$  and found a condition for the functor to map a finite-dimensional irreducible  $gl(n, \mathbb{F})$ -module to an infinite-dimensional irreducible  $sl(n+1, \mathbb{F})$ -module. Our general frame also gave a direct polynomial extension from irreducible  $gl(n, \mathbb{F})$ -modules to irreducible  $sl(n+1, \mathbb{F})$ -modules.

The work [14] lead to a one-parameter family of inhomogeneous first-order differential operator (oscillator) representations of  $sl(n+1, \mathbb{F})$ . By partially swapping differential operators and multiplication operators, we obtain more general differential operator (oscillator) representations of  $sl(n+1, \mathbb{F})$ . In this paper, we construct new multi-parameter families of explicit infinite-dimensional irreducible representations for  $s(n+1, \mathbb{F})$  and  $sp(2m+2)$  when  $n = 2m+1$  by letting these differential operators act on the corresponding polynomial algebra and the space of exponential-polynomial functions. Some of the corresponding modules are explicit infinite-dimensional irreducible weight modules with finite-dimensional weight subspaces. Our results can be viewed as extensions of Howe's oscillator construction of infinite-dimensional multiplicity-free irreducible representations for  $sl(n, \mathbb{F})$  (cf. [11]). Indeed, Howe's result plays an important role in proving the irreducibility of the representations for  $sl(n+1, \mathbb{F})$ . The results on symplectic Lie algebras in this paper can be used to study the irreducible representations of the other simple Lie algebra via Howe's theta correspondence technique.

Let  $E_{r,s}$  be the  $(n+1) \times (n+1)$  matrix with 1 as its  $(r, s)$ -entry and 0 as the others. The special linear algebra

$$sl(n+1, \mathbb{F}) = \sum_{1 \leq i < j \leq n+1} (\mathbb{F}E_{i,j} + \mathbb{F}E_{j,i}) + \sum_{r=1}^n \mathbb{F}(E_{r,r} - E_{r+1,r+1}). \quad (1.3)$$

For any two integers  $p \leq q$ , we denote  $\overline{p, q} = \{p, p+1, \dots, q\}$ . Set  $D = \sum_{s=1}^n x_s \partial_{x_s}$ . According to Zhao and the author's work [14], we have the following one-parameter generalization  $\pi_c$  of the projective representation of  $sl(n+1, \mathbb{F})$ :

$$\pi_c(E_{i,j}) = x_i \partial_{x_j}, \quad \pi_c(E_{i,n+1}) = x_i(D+c), \quad \pi_c(E_{n+1,i}) = -\partial_{x_i}, \quad (1.4)$$

$$\pi_c(E_{i,i} - E_{j,j}) = x_i \partial_{x_i} - x_j \partial_{x_j}, \quad \pi_c(E_{n,n} - E_{n+1,n+1}) = D+c+x_n \partial_{x_n} \quad (1.5)$$

for  $i, j \in \overline{1, n}$  with  $i \neq j$ , where  $c \in \mathbb{F}$ .

Let  $S$  be a subset of  $\overline{1, n}$ . Note the symmetry:

$$[\partial_{x_r}, x_r] = 1 = [-x_r, \partial_{x_r}]. \quad (1.6)$$

Changing operators  $\partial_{x_r} \mapsto -x_r$  and  $x_r \mapsto \partial_{x_r}$  for  $r \in S$  in (1.4) and (1.5), we get another differential-operator representation  $\pi_{c,S}$  of  $sl(n+1, \mathbb{F})$ . We treat  $\pi_{c,\emptyset} = \pi_c$  and call  $\pi_{c,S}$  *projective oscillator representations* in terms of physics terminology. For  $\vec{a} = (a_1, a_2, \dots, a_n)^t \in \mathbb{F}^n$ , we denote  $\vec{a} \cdot \vec{x} = \sum_{i=1}^n a_i x_i$ . Let  $\mathcal{A} = \mathbb{F}[x_1, x_2, \dots, x_n]$  be the algebra of polynomials in  $x_1, x_2, \dots, x_n$ . Moreover, we set

$$\mathcal{A}_{\vec{a}} = \{f e^{\vec{a} \cdot \vec{x}} \mid f \in \mathcal{A}\}. \quad (1.7)$$

Denote by  $\pi_{c,S}^{\vec{a}}$  the representation  $\pi_{c,S}$  of  $sl(n+1, \mathbb{F})$  on  $\mathcal{A}_{\vec{a}}$  and by  $\mathbb{N}$  the set of nonnegative integers. In [14], Zhao and the author proved that the representation  $\pi_{c,\emptyset}^{\vec{0}}$  of  $sl(n+1, \mathbb{F})$  is irreducible if and only if  $c \notin -\mathbb{N}$ . Moreover,  $\mathcal{A}$  has a composite series of length 2 when  $c \in -\mathbb{N}$ . In this paper, we prove:

**Theorem 1.** *Let  $S$  be a proper subset of  $\overline{1, n}$ . The representation  $\pi_{c,S}^{\vec{0}}$  is irreducible for any  $c \in \mathbb{F} \setminus \mathbb{Z}$ , and the underlying module  $\mathcal{A}$  is an infinite-dimensional weight  $sl(n+1, \mathbb{F})$ -module with finite-dimensional weight subspaces. If  $a_i \neq 0$  for some  $i \in \overline{1, n} \setminus S$  or  $|S| > 1$  and  $\vec{a} \neq \vec{0}$ , then the representation  $\pi_{c,S}^{\vec{a}}$  of  $sl(n+1, \mathbb{F})$  is always irreducible for any  $c \in \mathbb{F}$ .*

Suppose that  $n = 2m + 1 > 1$  is an odd integer and the subset  $S$  satisfies:

$$m+1 \notin S \text{ and for } i \in \overline{1, m}, \text{ at most one of } i \text{ and } i+m+1 \text{ in } S. \quad (1.8)$$

Our second main theorem in this paper is as follows.

**Theorem 2.** *If  $c \notin -\mathbb{N}$ , the restricted representation  $\pi_{c,\emptyset}^{\vec{0}}$  of  $sp(2m+2, \mathbb{F})$  is irreducible. When  $c \in -\mathbb{N}$ , the  $sp(2m+2, \mathbb{F})$ -module  $\mathcal{A}$  has a composite series of length 2 with respect to the restricted representation  $\pi_{c,\emptyset}^{\vec{0}}$ .*

*The restricted representation  $\pi_{c,S}^{\vec{0}}$  of  $sp(2m+2, \mathbb{F})$  with  $S \neq \emptyset$  is irreducible for any  $c \in \mathbb{F} \setminus \mathbb{Z}$ . Suppose that  $\vec{a} \neq \vec{0}$ ,  $a_{m+1} = 0$ ,  $a_{i_0} \neq 0$  for some  $m+1+i_0 \in S \cap \overline{m+2, 2m+1}$  if  $S \cap \overline{m+2, 2m+1} \neq \emptyset$ , and  $a_{m+1+j_0} \neq 0$  for some  $j_0 \in S \cap \overline{1, m+1}$  if  $S \cap \overline{1, m+1} \neq \emptyset$ , then the restricted representation  $\pi_{c,S}^{\vec{a}}$  of  $sp(2m+2, \mathbb{F})$  is irreducible for any  $c \in \mathbb{F}$ .*

*With respect to the restricted representation  $\pi_{c,S}^{\vec{0}}$ ,  $\mathcal{A}$  is an infinite-dimensional weight  $sp(2m+2, \mathbb{F})$ -module with finite-dimensional weight subspaces.*

In Section 2, we prove Theorem 1. The proof of Theorem 2 is given in Section 3.

## 2 Proof of Theorem 1

In this section, we will prove Theorem 1 case by case.

*Case 1. The representation  $\pi_{c,S}^{\vec{0}}$  with  $S \neq \emptyset, \overline{1, n}$ .*

Without loss of generality, we assume  $S = \overline{1, n_1}$  for some  $n_1 \in \overline{1, n_1}$  and  $n_1 < n$ . Set

$$\tilde{D} = \sum_{r=n_1+1}^n x_r \partial_{x_r} - \sum_{i=1}^{n_1} x_i \partial_{x_i}. \quad (2.1)$$

Then the representation  $\pi_{c,S}^{\vec{0}}$  of  $sl(n+1, \mathbb{F})$  is the representation  $\pi_{c,S}$  on  $\mathcal{A}$  with

$$\pi_{c,S}(E_{i,j}) = \begin{cases} -x_j \partial_{x_i} - \delta_{i,j} & \text{if } i, j \in \overline{1, n_1}; \\ \partial_{x_i} \partial_{x_j} & \text{if } i \in \overline{1, n_1}, j \in \overline{n_1+1, n}; \\ -x_i x_j & \text{if } i \in \overline{n_1+1, n}, j \in \overline{1, n_1}; \\ x_i \partial_{x_j} & \text{if } i, j \in \overline{n_1+1, n}, \end{cases} \quad (2.2)$$

$$\pi_{c,S}(E_{i,n+1}) = \begin{cases} (\tilde{D} + c - n_1 - 1) \partial_{x_i} & \text{if } i \leq n_1, \\ x_i (\tilde{D} + c - n_1) & \text{if } i > n_1, \end{cases} \quad (2.3)$$

$$\pi_{c,S}(E_{n+1,i}) = \begin{cases} x_i & \text{if } i \leq n_1, \\ -\partial_{x_i} & \text{if } i > n_1, \end{cases} \quad (2.4)$$

$$\pi_{c,S}(E_{n,n} - E_{n+1,n+1}) = \tilde{D} - n_1 + c + x_n \partial_{x_n} \quad (2.5)$$

For any  $k \in \mathbb{Z}$ , we denote

$$\mathcal{A}_{\langle k \rangle} = \text{Span} \left\{ x^\alpha = \prod_{i=1} x_i^{\alpha_i} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n; \sum_{i=1}^{n_1} \alpha_i - \sum_{r=n_1+1}^n \alpha_r = k \right\}. \quad (2.6)$$

Then  $\mathcal{A} = \bigoplus_{k \in \mathbb{Z}} \mathcal{A}_{\langle k \rangle}$  and

$$\mathcal{A}_{\langle k \rangle} = \{ f \in \mathcal{A} \mid \tilde{D}(f) = kf \}. \quad (2.7)$$

Note that

$$\mathcal{G}_0 = \sum_{1 \leq i < j \leq n} (\mathbb{F}E_{i,j} + \mathbb{F}E_{j,i}) + \sum_{r=1}^{n-1} \mathbb{F}(E_{r,r} - E_{r+1,r+1}) \quad (2.8)$$

is a Lie subalgebra of  $sl(n+1, \mathbb{F})$  isomorphic to  $sl(n, \mathbb{F})$ . The following result was due to Howe [11].

**Lemma 2.1.** *Let  $\ell_1, \ell_2 \in \mathbb{N}$  with  $\ell_1 > 0$ ,  $\mathcal{A}_{\langle -\ell_1 \rangle}$  is an irreducible highest-weight  $\mathcal{G}_0$ -submodule with highest weight  $\ell_1 \lambda_{n_1-1} - (\ell_1 + 1) \lambda_{n_1}$  and  $\mathcal{A}_{\langle \ell_2 \rangle}$  is an irreducible highest-weight  $\mathcal{G}_0$ -submodule with highest weight  $-(\ell_2 + 1) \lambda_{n_1} + \ell_2 \lambda_{n_1+1}$ .*

Now we have the first result in this section.

**Theorem 2.2.** *The representation  $\pi_{c,S}^{\vec{0}}$  of  $sl(n+1, \mathbb{F})$  is irreducible if any  $c \notin \mathbb{Z}$ .*

*Proof.* Let  $k$  be any integer. For any  $0 \neq f \in \mathcal{A}_{\langle k \rangle}$ , we have

$$0 \neq E_{n+1,1}(f) = x_1 f \in \mathcal{A}_{\langle k-1 \rangle} \quad (2.9)$$

by (2.4), and

$$0 \neq E_{n,n+1}(f) = (k + c - n_1)x_n f \in \mathcal{A}_{\langle k+1 \rangle} \quad (2.10)$$

by (2.3). Let  $\mathcal{M}$  be a nonzero  $sl(n+1, \mathbb{F})$ -submodule of  $\mathcal{A}$ . If  $k_1, k_2 \in \mathbb{Z}$  with  $k_1 \neq k_2$ , then the highest weights of  $\mathcal{A}_{\langle k_1 \rangle}$  and  $\mathcal{A}_{\langle k_2 \rangle}$  are different as  $\mathcal{G}_0$ -modules by Lemma 2.1. So  $\mathcal{A}_{\langle k_0 \rangle} \subset \mathcal{M}$  for some  $k_0 \in \mathbb{Z}$ . Moreover, (2.9) and (2.10) imply  $\mathcal{A}_{\langle k \rangle} \subset \mathcal{M}$  for any  $k \in \mathbb{Z}$ . Hence  $\mathcal{M} = \mathcal{A}$ .  $\square$

Expressions (2.2)-(2.5) imply the above representation is not of highest-weight type. Moreover,  $\mathcal{A}$  is a weight  $sl(n+1, \mathbb{F})$ -module with finite-dimensional weight subspaces.

*Case 2. The representation  $\pi_{c,\emptyset}^{\vec{a}}$  with  $\vec{0} \neq \vec{a} \in \mathbb{F}^n$ .*

In this case,

$$E_{n+1,i}(f e^{\vec{a} \cdot \vec{x}}) = -(\partial_{x_i} + a_i)(f) e^{\vec{a} \cdot \vec{x}} \quad \text{for } i \in \overline{1, n}, f \in \mathcal{A}. \quad (2.11)$$

Thus

$$(E_{n+1,i} + a_i)(f e^{\vec{a} \cdot \vec{x}}) = -\partial_{x_i}(f) e^{\vec{a} \cdot \vec{x}} \quad \text{for } i \in \overline{1, n}, f \in \mathcal{A}. \quad (2.12)$$

The second result in this section.

**Theorem 2.3.** *The representation  $\pi_{c,\emptyset}^{\vec{a}}$  with  $\vec{0} \neq \vec{a} \in \mathbb{F}^n$  is an irreducible representation of  $sl(n+1, \mathbb{F})$  for any  $c \in \mathbb{F}$ .*

*Proof.* Let  $\mathcal{A}_k$  be the subspace of homogeneous polynomials with degree  $k$ . Set

$$\mathcal{A}_{\vec{a},k} = \mathcal{A}_k e^{\vec{a} \cdot \vec{x}} \quad \text{for } k \in \mathbb{N}. \quad (2.13)$$

Without loss of generality, we assume  $a_1 \neq 0$ . Let  $\mathcal{M}$  be a nonzero  $sl(n+1, \mathbb{F})$ -submodule of  $\mathcal{A}_{\vec{a}}$ . Take any  $0 \neq fe^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$  with  $f \in \mathcal{A}$ . By (2.12),

$$\partial_{x_i}(f)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{1, n}. \quad (2.14)$$

By induction, we have  $e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$ ; that is,  $\mathcal{A}_{\vec{a}, 0} \subset \mathcal{M}$ .

Suppose  $\mathcal{A}_{\vec{a}, \ell} \subset \mathcal{M}$  for some  $\ell \in \mathbb{N}$ . For any  $ge^{\vec{a} \cdot \vec{x}} \in \mathcal{A}_{\vec{a}, \ell}$ ,

$$E_{i,1}(ge^{\vec{a} \cdot \vec{x}}) = x_i(\partial_{x_1} + a_1)(g)e^{\vec{a} \cdot \vec{x}} = a_1x_ig e^{\vec{a} \cdot \vec{x}} + x_i\partial_{x_1}(g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{2, n} \quad (2.15)$$

by (1.4). Since  $x_i\partial_{x_1}(g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{A}_{\vec{a}, \ell} \subset \mathcal{M}$ , we have

$$x_ig e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{2, n}. \quad (2.16)$$

On the other hand,

$$(E_{1,1} - E_{2,2})(ge^{\vec{a} \cdot \vec{x}}) = a_1x_1ge^{\vec{a} \cdot \vec{x}} + (x_1\partial_{x_1} - x_2\partial_{x_2} - a_2x_2)(g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (2.17)$$

by (1.5). Our assumption says that  $(x_1\partial_{x_1} - x_2\partial_{x_2})(g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{A}_{\vec{a}, \ell} \subset \mathcal{M}$ . According to (2.16),  $-a_2x_2(g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$ . Therefore,

$$x_1ge^{\vec{a} \cdot \vec{x}} \in \mathcal{M}. \quad (2.18)$$

Expressions (2.17) and (2.18) imply  $\mathcal{A}_{\vec{a}, \ell+1} \subset \mathcal{M}$ . By induction,  $\mathcal{A}_{\vec{a}, \ell} \subset \mathcal{M}$  for any  $\ell \in \mathbb{N}$ . So  $\mathcal{A}_{\vec{a}} = \mathcal{M}$ . Hence  $\mathcal{A}_{\vec{a}}$  is an irreducible  $sl(n+1, \mathbb{F})$ -module.  $\square$

*Case 3. The representation  $\pi_{c,S}^{\vec{a}}$  with  $a_i \neq 0$  for some  $i \in \overline{1, n} \setminus S$  or  $|S| > 1$ .*

The following is the third result in this section.

**Theorem 2.4.** *Under the above assumption, the representation  $\pi_{c,S}^{\vec{a}}$  with  $\vec{0} \neq \vec{a} \in \mathbb{F}^n$  is an irreducible representation of  $sl(n+1, \mathbb{F})$ .*

*Proof.* Without loss of generality, we assume  $S = \overline{1, n_1}$  for some  $n_1 \in \overline{1, n_1}$  and  $n_1 < n$ . Let  $\mathcal{M}$  be a nonzero  $sl(n+1, \mathbb{F})$ -submodule of  $\mathcal{A}_{\vec{a}}$ . By (2.4) and (2.11)-(2.14), there exists  $0 \neq fe^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$  with  $f \in \mathbb{F}[x_1, \dots, x_{n_1}]$ .

*Subcase (1).  $a_i \neq 0$  for some  $i \in \overline{n_1+1, n}$ .*

By symmetry, we can assume  $a_n \neq 0$ . According to (2.2),

$$E_{i,n}(fe^{\vec{a} \cdot \vec{x}}) = (\partial_{x_i} + a_i)(\partial_{x_n} + a_n)(f)e^{\vec{a} \cdot \vec{x}} = a_ia_nfe^{\vec{a} \cdot \vec{x}} + a_n\partial_{x_i}(f)e^{\vec{a} \cdot \vec{x}} \quad \text{for } i \in \overline{1, n_1}. \quad (2.19)$$

Thus

$$(a_n^{-1}E_{i,n} - a_i)(fe^{\vec{a} \cdot \vec{x}}) = \partial_{x_i}(f)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{1, n_1}. \quad (2.20)$$

By induction on the degree of  $f$ , we get  $e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$ ; that is,  $\mathcal{A}_{\vec{a}, 0} \subset \mathcal{M}$ .

The arguments in (2.15)-(2.18) yield

$$\mathbb{F}[x_{n_1+1}, \dots, x_n]e^{\vec{a} \cdot \vec{x}} \subset \mathcal{M}. \quad (2.21)$$

According to (2.4),

$$E_{n+1,1}^{\ell_1} \cdots E_{n+1,n_1}^{\ell_{n_1}}(\mathbb{F}[x_{n_1+1}, \dots, x_n]e^{\vec{a} \cdot \vec{x}}) = x_1^{\ell_1} \cdots x_{n_1}^{\ell_{n_1}}(\mathbb{F}[x_{n_1+1}, \dots, x_n]e^{\vec{a} \cdot \vec{x}}) \subset \mathcal{M} \quad (2.22)$$

for  $\ell_i \in \mathbb{N}$  with  $i \in \overline{1, n_1}$ . Thus  $\mathcal{A}_{\vec{a}} = \mathcal{M}$ . So  $\mathcal{A}_{\vec{a}}$  is an irreducible  $sl(n+1, \mathbb{F})$ -module.

*Subcase (2).  $a_i = 0$  for any  $i \in \overline{n_1+1, n}$  and  $n_1 > 1$ .*

By the transformation

$$\vec{x} \mapsto T\vec{x}, \quad A \mapsto TAT^{-1} \quad (2.23)$$

with  $A \in sl(n+1, \mathbb{F})$  for some  $n \times n$  orthogonal matrix  $T$ , we can assume  $a_1 \neq 0$  and  $a_i = 0$  for  $i \in \overline{2, n}$ . Note that

$$E_{1,2}(fe^{\vec{a} \cdot \vec{x}}) = -x_2(\partial_{x_1} + a_1)(f)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (2.24)$$

by (2.2) and

$$E_{n+1,2}(fe^{\vec{a} \cdot \vec{x}}) = x_2 fe^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (2.25)$$

by (2.4). Thus

$$x_2 \partial_{x_1}(f)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}. \quad (2.26)$$

Repeatedly applying (2.26) if necessary, we can assume  $f \in \mathbb{F}[x_2, \dots, x_n]$ . We apply the arguments in the proof of Theorem 2.2 to the Lie subalgebra

$$\mathcal{L} = \sum_{2 \leq i < j \leq n+1} (\mathbb{F}E_{i,j} + \mathbb{F}E_{j,i}) + \sum_{r=2}^n \mathbb{F}(E_{r,r} - E_{r+1,r+1}) \quad (2.27)$$

and obtain

$$\mathbb{F}[x_2, \dots, x_n]e^{\vec{a} \cdot \vec{x}} \subset \mathcal{M}. \quad (2.28)$$

According to (2.4),

$$E_{n+1,1}^\ell(\mathbb{F}[x_2, \dots, x_n]e^{\vec{a} \cdot \vec{x}}) = x_1^\ell(\mathbb{F}[x_2, \dots, x_n]e^{\vec{a} \cdot \vec{x}}) \subset \mathcal{M} \quad (2.29)$$

for any  $\ell \in \mathbb{N}$ . Therefore  $\mathcal{M} = \mathcal{A}_{\vec{a}}$ . So  $\mathcal{A}_{\vec{a}}$  is an irreducible  $sl(n+1, \mathbb{F})$ -module.  $\square$

With the representation  $\pi_{c,S}^{\vec{0}}$ ,  $\mathcal{A}$  is an infinite-dimensional weight  $sl(n+1, \mathbb{F})$ -module with finite-dimensional weight subspaces by (2.2) and (2.5). Now Theorem 1 follows from Theorems 2.2, 2.3 and 2.4.

### 3 Proof of Theorem 2

Assume that  $n = 2m + 1 > 1$  is an odd integer. In this section, we will give the proof of Theorem 2.

Recall that the symplectic Lie algebras

$$\begin{aligned} sp(2m+2, \mathbb{F}) &= \sum_{1 \leq r \leq s \leq m+1} [\mathbb{F}(E_{r,m+1+s} + E_{s,m+1+r}) + \mathbb{F}(E_{m+1+r,s} + E_{m+1+s,r})] \\ &\quad + \sum_{i,j=1}^{m+1} \mathbb{F}(E_{i,j} - E_{m+1+j,m+1+i}). \end{aligned} \quad (3.1)$$

For convenience, we rednote

$$x_0 = x_{m+1}, \quad y_i = x_{m+1+i} \quad \text{for } i \in \overline{1, m}. \quad (3.2)$$

In particular,

$$D = \sum_{i=0}^m x_i \partial_{x_i} + \sum_{r=1}^m y_r \partial_{y_r}. \quad (3.3)$$

According to (1.4) and (1.5), we have the representation  $\pi_c$  of  $sp(2m+2, \mathbb{F})$ :

$$\pi_c(E_{i,j} - E_{m+1+j,m+1+i}) = x_i \partial_{x_j} - y_j \partial_{y_i}, \quad \pi_c(E_{i,m+1+j} + E_{j,m+1+i}) = x_i \partial_{y_j} + x_j \partial_{y_i}, \quad (3.4)$$

$$\pi_c(E_{2m+2,m+1}) = -\partial_{x_0}, \quad \pi_c(E_{m+1,i} - E_{m+1+i,2m+2}) = x_0\partial_{x_i} - y_i(D+c), \quad (3.5)$$

$$\pi_c(E_{i,m+1} - E_{2m+2,m+1+i}) = x_i\partial_{x_0} + \partial_{y_i}, \quad \pi_c(E_{2m+2,i} + E_{m+1+i,m+1}) = y_i\partial_{x_0} - \partial_{x_i}, \quad (3.6)$$

$$\pi_c(E_{m+1+i,j} + E_{m+1+j,i}) = y_i\partial_{x_j} + y_j\partial_{x_i}, \quad \pi_c(E_{m+1,m+1} - E_{2m+2,2m+2}) = D + x_0\partial_{x_0} + c, \quad (3.7)$$

$$\pi_c(E_{m+1,m+1+i} + E_{i,2m+2}) = x_0\partial_{y_i} + x_i(D+c), \quad \pi_c(E_{i,i} - E_{m+i,m+i}) = x_i\partial_{x_i} - y_i\partial_{y_i}, \quad (3.8)$$

$$\pi_c(E_{m+1,2m+2}) = x_0(D+c) \quad (3.9)$$

for  $i, j \in \overline{1, m}$ .

Denote

$$\begin{aligned} \mathcal{K} = & \sum_{1 \leq r \leq s \leq m} [\mathbb{F}(E_{r,m+1+s} + E_{s,m+1+r}) + \mathbb{F}(E_{m+1+r,s} + E_{m+1+s,r})] \\ & + \sum_{i,j=1}^m \mathbb{F}(E_{i,j} - E_{m+1+j,m+1+i}), \end{aligned} \quad (3.10)$$

which is a Lie subalgebra of  $sp(2m+2, \mathbb{F})$  isomorphic to  $sp(2m, \mathbb{F})$ . We will prove Theorem 2 case by case.

*Case 1.*  $\vec{a} = \vec{0}$  and  $S = \emptyset$ .

Let  $\mathcal{B} = \mathbb{F}[x_1, \dots, x_n, y_1, \dots, y_n]$ . Denote by  $\mathcal{B}_k$  the subspace of homogeneous polynomials with degree  $k$ . First we have the following well-known result (e.g., cf. [7]).

**Lemma 3.1.** *For any  $k \in \mathbb{N}$ ,  $\mathcal{B}_k$  forms a finite-dimensional irreducible  $\mathcal{K}$ -module with highest weight  $k\lambda_1$ .*

Set

$$\mathcal{A}(\ell) = \sum_{i=0}^{\ell} \mathcal{A}_i \quad \text{for } \ell \in \mathbb{N}. \quad (3.11)$$

Take the Cartan subalgebra

$$H = \sum_{i=1}^{m+1} \mathbb{F}(E_{i,i} - E_{m+1+i,m+1+i}) \quad (3.12)$$

of  $sp(2m+2, \mathbb{F})$ . Define  $\{\varepsilon_1, \dots, \varepsilon_{n+1}\} \subset H^*$  by:

$$\varepsilon_j(E_{i,i} - E_{n+1+i,n+1+i}) = \delta_{i,j}. \quad (3.13)$$

Recall that the representation  $\pi_{c,\emptyset}^{\vec{0}}$  of  $sp(2m+2, \mathbb{F})$  is the representation  $\pi_c$  (cf. (3.4)-(3.9)) on the space  $\mathcal{A} = \mathbb{F}[x_0, \dots, x_m, y_1, \dots, y_m]$ . Then we have:

**Theorem 3.2.** *If  $c \notin -\mathbb{N}$ , the representation  $\pi_{c,\emptyset}^{\vec{0}}$  of  $sp(2m+2, \mathbb{F})$  given in (3.4)-(3.9) is a highest-weight irreducible representation with highest-weight  $-c\lambda_1$ . When  $-c = \ell \in \mathbb{N}$ ,  $\mathcal{A}_{(\ell)}$  is a finite-dimensional irreducible  $sp(2m+2, \mathbb{F})$ -module with highest weight  $\ell\lambda_n$  and  $\mathcal{A}/\mathcal{A}_{(\ell)}$  is an irreducible highest weight  $sp(2m+2, \mathbb{F})$ -module with highest weight  $-(\ell+2)\lambda_1 + (\ell+1)\lambda_2$ , where  $\lambda_i$  is the  $i$ th fundamental weight of  $sp(2m+2, \mathbb{F})$ .*

*Proof.* Observe that

$$\mathcal{A}_k = \sum_{s=0}^k x_0^s \mathcal{B}_{k-s} \quad \text{for } k \in \mathbb{N}. \quad (3.14)$$

Let  $\mathcal{M}$  be a nonzero  $sp(2m+2, \mathbb{F})$ -submodule of  $\mathcal{A}$ . Take any  $0 \neq f \in \mathcal{M}$ . Repeatedly applying the first equation in (3.5) and (3.6) to  $f$ , we obtain  $1 \in \mathcal{M}$ . Note

$$(E_{m+1, 2m+2})^k(1) = [\prod_{r=0}^{k-1} (r+c)]x_0^k \in \mathcal{M} \quad (3.15)$$

by (3.9) and

$$(E_{1, m+1} - E_{2m+2, m+1+1})^s(x_0^k) = [\prod_{i=0}^{s-1} (k-i)x_0^{k-s}x_1^s] \quad \text{for } s \in \overline{1, k} \quad (3.16)$$

by the first equation in (3.6). Suppose  $c \notin -\mathbb{N}$ . Then (3.17) yields

$$x_0^k \in \mathcal{M} \quad \text{for } k \in \mathbb{N}. \quad (3.17)$$

Moreover, (3.16) with  $k = r + s$  gives

$$x_0^r x_1^s \in V \quad \text{for } r, s \in \mathbb{N}. \quad (3.18)$$

Furthermore,

$$U(\mathcal{K})(x_0^r x_1^s) = x_0^r \mathcal{B}_s \subset \mathcal{M} \quad (3.19)$$

by Lemma 3.1. Thus

$$\mathcal{A} = \sum_{r, s=0}^{\infty} x_0^r \mathcal{B}_s \subset \mathcal{M}; \quad (3.20)$$

that is,  $\mathcal{M} = \mathcal{A}$ . So  $\mathcal{A}$  is an irreducible  $sp(2m+2, \mathbb{F})$ -module and 1 is its highest-weight vector with weight  $-c\lambda_1$  with respect to the following simple positive roots

$$\{\varepsilon_{n+1} - \varepsilon_n, \varepsilon_n - \varepsilon_{n-1}, \dots, \varepsilon_2 - \varepsilon_1, 2\varepsilon_1\}. \quad (3.21)$$

Next we assume  $c = -\ell$  with  $\ell \in \mathbb{N}$ . Since

$$E_{m+1, 2m+2}|_{\mathcal{A}_\ell} = 0, \quad (E_{m+1, i} - E_{m+1+i, 2m+2})|_{\mathcal{A}_\ell} = x_0 \partial_{x_i}, \quad (3.22)$$

$$(E_{m+1, m+1+i} + E_{i, 2m+2})|_{\mathcal{A}_\ell} = x_0 \partial_{y_i} \quad (3.23)$$

by (3.5), (3.8) and (3.9),  $\mathcal{A}_{(\ell)}$  is a finite-dimensional  $sp(2m+2, \mathbb{F})$ -module. Let  $\mathcal{M}$  be a nonzero  $sp(2m+2, \mathbb{F})$ -submodule of  $\mathcal{A}_{(\ell)}$ . By (3.15),

$$x_0^k \in \mathcal{M} \quad \text{for } k \in \overline{0, \ell}. \quad (3.24)$$

Moreover, (3.16) with  $k = r + s$  gives

$$x_0^r x_1^s \in \mathcal{M} \quad \text{for } r, s \in \overline{0, \ell} \text{ such that } r + s \leq \ell. \quad (3.25)$$

Thus

$$\mathcal{A}_{(\ell)} = \sum_{r=0}^{\ell} \sum_{s=0}^{\ell-r} x_0^r \mathcal{B}_s \subset \mathcal{M} \quad (3.26)$$

by Lemma 3.1; that is,  $\mathcal{M} = \mathcal{A}_{(\ell)}$ . So  $\mathcal{A}_{(\ell)}$  is an irreducible  $sp(2n+2, \mathbb{F})$ -module and 1 is again its highest-weight vector.

Consider the quotient  $sp(2n+2, \mathbb{F})$ -module  $\mathcal{A}/\mathcal{A}_{(\ell)}$ . Let  $W \supset \mathcal{A}_{(\ell)}$  be an  $sp(2n+2, \mathbb{F})$ -submodule of  $\mathcal{A}$  such that  $W \neq \mathcal{A}_{(\ell)}$ . Take any  $f \in W \setminus \mathcal{A}_{(\ell)}$ . Repeatedly applying (3.6)



and the first equation in (3.5) to  $f$  if necessary, we can assume  $f \in \mathcal{B}_{\ell+1}$ . Since  $\mathcal{B}_{\ell+1}$  is an irreducible  $\mathcal{K}$ -module, we have

$$\mathcal{B}_{\ell+1} \subset W. \quad (3.27)$$

In particular,  $x_1^{\ell+1} \in W$ . According to (3.8),

$$(E_{m+1,m+2} + E_{1,2m+2})^r(x_1^{\ell+1}) = r!x_1^{\ell+1+r} \in W \quad \text{for } 0 < r \in \mathbb{Z}. \quad (3.28)$$

Since  $\mathcal{B}_{\ell+1+r} \ni x_1^{\ell+1+r}$  is an irreducible  $\mathcal{K}$ -module, we have

$$\mathcal{B}_{\ell+1+r} \subset W. \quad (3.29)$$

Suppose that

$$x_0^r \mathcal{B}_s \subset W \quad \text{for } r \in \overline{0, k} \text{ and } s \in \mathbb{N} \text{ such that } r + s \geq \ell + 1. \quad (3.30)$$

Fix such  $r$  and  $s$ . Observe  $x_0^r x_1^{s-1} y_1 \in x_0^r \mathcal{B}_s \subset W$ . Using the first equation in (3.8), we get

$$(E_{m+1,m+2} + E_{1,2m+2})(x_0^r x_1^{s-1} y_1) = (r + s - \ell)x_0^r x_1^s y_1 + x_0^{r+1} x_1^{s-1} \in W. \quad (3.31)$$

By the assumption (3.30),  $(r + s - \ell)x_0^r x_1^s y_1 \in x_0^r \mathcal{B}_{s+1} \subset W$ . So

$$x_0^{r+1} x_1^{s-1} \in W \cap x_0^{r+1} \mathcal{B}_{s-1}. \quad (3.32)$$

Since  $x_0^{r+1} \mathcal{B}_{s-1}$  is an irreducible  $\mathcal{K}$ -module, we get

$$x_0^{r+1} \mathcal{B}_{s-1} \subset W. \quad (3.33)$$

By induction on  $r$ , we prove

$$x_0^r \mathcal{B}_s \subset W \quad \text{for } r, s \in \mathbb{N} \text{ such that } r + s \geq \ell + 1. \quad (3.34)$$

According to (3.14),

$$\sum_{k=\ell+1}^{\infty} \mathcal{A}_k \subset W. \quad (3.35)$$

Since  $W \supset \mathcal{A}_{(\ell)}$ , we have  $W = \mathcal{A}$ . So  $\mathcal{A}/\mathcal{A}_{(\ell)}$  is an irreducible  $sp(2m+2, \mathbb{F})$ -module. Moreover,  $x_n^{\ell+1}$  is a highest weight vector of weight  $-(\ell+2)\lambda_1 + (\ell+1)\lambda_2$  with respect to (3.21).  $\square$

*Case 2.*  $\vec{a} \neq \vec{0}$ ,  $a_{m+1} = 0$  and  $S = \emptyset$ .

For simplicity, we redenote

$$b_i = a_{m+1+i} \quad \text{for } i \in \overline{1, m}. \quad (3.36)$$

Recall that the representation  $\pi_{c, \emptyset}^{\vec{a}}$  of  $sp(2m+2, \mathbb{F})$  is the representation  $\pi_c$  (cf. (3.4)-(3.9)) on the space  $\mathcal{A}_{\vec{a}}$  (cf. (1.7)). Our second result in this section is:

**Theorem 3.3.** *The representation  $\pi_{c, \emptyset}^{\vec{a}}$  with  $\vec{0} \neq \vec{a} \in \mathbb{F}^n$  and  $a_{m+1} = 0$  is an irreducible representation of  $sp(2m+2, \mathbb{F})$  for any  $c \in \mathbb{F}$ .*

*Proof.* By symmetry, we may assume  $a_1 \neq 0$ . Let  $\mathcal{M}$  be a nonzero  $sp(2m+2, \mathbb{F})$ -submodule of  $\mathcal{A}_{\vec{a}}$ . Take any  $0 \neq f e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$  with  $f \in \mathcal{A}$ . By the assumption  $a_0 = 0$ , (3.5) and (3.6),

$$E_{2m+2, m+1}(f e^{\vec{a} \cdot \vec{x}}) = -\partial_{x_0}(f) e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}, \quad (3.37)$$

$$(E_{i,m+1} - E_{2m+2,m+1+i} - b_i)(fe^{\vec{a}\cdot\vec{x}}) = [\partial_{y_i}(f) + x_i\partial_{x_0}(f)]e^{\vec{a}\cdot\vec{x}} \in \mathcal{M}, \quad (3.38)$$

$$(E_{2m+2,i} + E_{m+1+i,m+1} + a_i)(fe^{\vec{a}\cdot\vec{x}}) = [-\partial_{x_i}(f) + y_i\partial_{x_0}(f)]e^{\vec{a}\cdot\vec{x}} \in \mathcal{M} \quad (3.39)$$

for  $i \in \overline{1, m}$ . Repeatedly applying (3.37)-(3.39), we obtain  $e^{\vec{a}\cdot\vec{x}} \in \mathcal{M}$ . Equivalently,  $\mathcal{A}_{\vec{a},0} \subset \mathcal{M}$  (cf. (2.13)).

Suppose  $\mathcal{A}_{\vec{a},\ell} \subset \mathcal{M}$  for some  $\ell \in \mathbb{N}$ . For any  $ge^{\vec{a}\cdot\vec{x}} \in \mathcal{A}_{\vec{a},\ell}$ ,

$$(E_{i,1} - E_{m+2,m+1+i})(ge^{\vec{a}\cdot\vec{x}}) = [a_1x_i - b_iy_1 + x_i\partial_{x_1} - y_1\partial_{x_i}](g)e^{\vec{a}\cdot\vec{x}} \in \mathcal{M} \quad (3.40)$$

by (3.4) and

$$(E_{m+1+i,1} + E_{m+2,i})(ge^{\vec{a}\cdot\vec{x}}) = [a_1y_i + a_iy_1 + y_i\partial_{x_1} + y_1\partial_{x_i}](g)e^{\vec{a}\cdot\vec{x}} \in \mathcal{M} \quad (3.41)$$

by the first equation in (3.7), where  $i \in \overline{1, m}$ . Since

$$(x_i\partial_{x_1} - y_1\partial_{x_i})(g)e^{\vec{a}\cdot\vec{x}}, (y_i\partial_{x_1} + y_1\partial_{x_i})(g)e^{\vec{a}\cdot\vec{x}} \in \mathcal{A}_{\vec{a},\ell} \subset \mathcal{M}, \quad (3.42)$$

we have

$$(a_1x_i - b_iy_1)ge^{\vec{a}\cdot\vec{x}}, (a_1y_i + a_iy_1)ge^{\vec{a}\cdot\vec{x}} \in \mathcal{M} \quad (3.43)$$

for  $i \in \overline{1, m}$ . The above second equation with  $i = 1$  gives

$$2a_1y_1ge^{\vec{a}\cdot\vec{x}} \in \mathcal{M} \Rightarrow y_1ge^{\vec{a}\cdot\vec{x}} \in \mathcal{M}. \quad (3.44)$$

Thus (3.43) yields

$$x_iye^{\vec{a}\cdot\vec{x}}, y_iye^{\vec{a}\cdot\vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{1, m}. \quad (3.45)$$

According to the second equation in (3.5),

$$\begin{aligned} & (E_{m+1,1} - E_{m+2,2m+2})(ge^{\vec{a}\cdot\vec{x}}) \\ &= [a_1x_0 - \sum_{i=1}^m (a_ix_i + b_iy_i)y_1 + x_0\partial_{x_1} - y_1(D+c)](g)e^{\vec{a}\cdot\vec{x}} \in \mathcal{M}. \end{aligned} \quad (3.46)$$

Replacing  $ge^{\vec{a}\cdot\vec{x}} \in \mathcal{A}_{\vec{a},\ell}$  by  $ge^{\vec{a}\cdot\vec{x}} \in \sum_{i=1}^m (x_i\mathcal{A}_{\vec{a},\ell} + y_i\mathcal{A}_{\vec{a},\ell})$  in (3.40)-(3.45), we obtain

$$x_iy_1ge^{\vec{a}\cdot\vec{x}}, y_iy_1ge^{\vec{a}\cdot\vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{1, m}. \quad (3.47)$$

Since  $D(g) = \ell g$  and  $x_0\partial_{x_1}(g)e^{\vec{a}\cdot\vec{x}} \in \mathcal{A}_{\vec{a},\ell} \subset \mathcal{M}$ , we have

$$[-\sum_{i=1}^m (a_ix_i + b_iy_i)y_1 + x_0\partial_{x_1} - y_1(D+c)](g)e^{\vec{a}\cdot\vec{x}} \in \mathcal{M}. \quad (3.48)$$

Hence (3.46) yields  $x_0ge^{\vec{a}\cdot\vec{x}} \in \mathcal{M}$ . Therefore,  $\mathcal{A}_{\vec{a},\ell+1} \subset \mathcal{M}$ . By induction,  $\mathcal{A}_{\vec{a},\ell} \subset \mathcal{M}$  for any  $\ell \in \mathbb{N}$ . So  $\mathcal{A}_{\vec{a}} = \mathcal{M}$ . Hence  $\mathcal{A}_{\vec{a}}$  is an irreducible  $sp(2m+2, \mathbb{F})$ -module.  $\square$

*Case 3.*  $\vec{a} = \vec{0}$  and  $S \neq \emptyset$ .

By symmetry and the assumption (1.8), we can assume

$$S = \overline{1, m_1} \bigcup \overline{m_2+1, m}, \quad m_1, m_2 \in \overline{1, m} \text{ and } m_1 \leq m_2, \quad (3.49)$$

where we treat  $\overline{m+1, m} = \emptyset$  when  $m_2 = m$ . Set

$$\tilde{D} = x_0\partial_{x_0} + \sum_{r=m_1+1}^m x_r\partial_{x_r} - \sum_{i=1}^{m_1} x_i\partial_{x_i} + \sum_{i=1}^{m_2} y_i\partial_{y_i} - \sum_{r=m_2+1}^m y_r\partial_{y_r} \quad (3.50)$$

and

$$\tilde{c} = c + m_2 - m_1 - m. \quad (3.51)$$

Then we have the following representation  $\pi_{c,S}$  of the Lie algebra  $sp(2m+2, \mathbb{F})$  determined by

$$\pi_{c,S}(E_{i,j} - E_{m+1+j,m+1+i}) = E_{i,j}^x - E_{j,i}^y \quad (3.52)$$

with

$$E_{i,j}^x = \begin{cases} -x_j \partial_{x_i} - \delta_{i,j} & \text{if } i, j \in \overline{1, m_1}; \\ \partial_{x_i} \partial_{x_j} & \text{if } i \in \overline{1, m_1}, j \in \overline{m_1 + 1, m}; \\ -x_i x_j & \text{if } i \in \overline{m_1 + 1, m}, j \in \overline{1, m_1}; \\ x_i \partial_{x_j} & \text{if } i, j \in \overline{m_1 + 1, m} \end{cases} \quad (3.53)$$

and

$$E_{i,j}^y = \begin{cases} y_i \partial_{y_j} & \text{if } i, j \in \overline{1, m_2}; \\ -y_i y_j & \text{if } i \in \overline{1, m_2}, j \in \overline{m_2 + 1, m}; \\ \partial_{y_i} \partial_{y_j} & \text{if } i \in \overline{m_2 + 1, m}, j \in \overline{1, m_2}; \\ -y_j \partial_{y_i} - \delta_{i,j} & \text{if } i, j \in \overline{m_2 + 1, m}, \end{cases} \quad (3.54)$$

and

$$\pi_{c,S}(E_{i,m+1+j}) = \begin{cases} \partial_{x_i} \partial_{y_j} & \text{if } i \in \overline{1, m_1}, j \in \overline{1, m_2}, \\ -y_j \partial_{x_i} & \text{if } i \in \overline{1, m_1}, j \in \overline{m_2 + 1, m}, \\ x_i \partial_{y_j} & \text{if } i \in \overline{m_1 + 1, m}, j \in \overline{1, m_2}, \\ -x_i y_j & \text{if } i \in \overline{m_1 + 1, m}, j \in \overline{m_2 + 1, m}, \end{cases} \quad (3.55)$$

$$\pi_{c,S}(E_{m+1+i,j}) = \begin{cases} -x_j y_i & \text{if } j \in \overline{1, m_1}, i \in \overline{1, m_2}, \\ -x_j \partial_{y_i} & \text{if } j \in \overline{1, m_1}, i \in \overline{m_2 + 1, m}, \\ y_i \partial_{x_j} & \text{if } j \in \overline{m_1 + 1, m}, i \in \overline{1, m_2}, \\ \partial_{x_j} \partial_{y_i} & \text{if } j \in \overline{m_1 + 1, m}, i \in \overline{m_2 + 1, m}, \end{cases} \quad (3.56)$$

$$\pi_{c,S}(E_{2m+2,m+1}) = -\partial_{x_0}, \quad \pi_{c,S}(E_{2m+2,m+1}) = x_0(\tilde{D} + \tilde{c}), \quad (3.57)$$

$$\pi_{c,S}(E_{i,m+1} - E_{2m+2,m+1+i}) = \begin{cases} \partial_{x_0} \partial_{x_i} + \partial_{y_i} & \text{if } i \in \overline{1, m_1}, \\ x_i \partial_{x_0} + \partial_{y_i} & \text{if } i \in \overline{m_1 + 1, m_2}, \\ x_i \partial_{x_0} - y_i & \text{if } i \in \overline{m_2 + 1, m}, \end{cases} \quad (3.58)$$

$$\pi_{c,S}(E_{2m+2,i} + E_{m+1+i,m+1}) = \begin{cases} y_i \partial_{x_0} + x_i & \text{if } i \in \overline{1, m_1}, \\ y_i \partial_{x_0} - \partial_{x_i} & \text{if } i \in \overline{m_1 + 1, m_2}, \\ \partial_{x_0} \partial_{y_i} - \partial_{x_i} & \text{if } i \in \overline{m_2 + 1, m}, \end{cases} \quad (3.59)$$

$$\pi_{c,S}(E_{m+1,i} - E_{m+1+i,2m+2}) = \begin{cases} -x_0 x_i - y_i(\tilde{D} + \tilde{c}) & \text{if } i \in \overline{1, m_1}, \\ x_0 \partial_{x_i} - y_i(\tilde{D} + \tilde{c}) & \text{if } i \in \overline{m_1 + 1, m_2}, \\ x_0 \partial_{x_i} - (\tilde{D} + \tilde{c} - 1) \partial_{y_i} & \text{if } i \in \overline{m_2 + 1, m}, \end{cases} \quad (3.60)$$

$$\pi_{c,S}(E_{m+1,m+1+i} + E_{i,2m+2}) = \begin{cases} x_0 \partial_{y_i} + (\tilde{D} + \tilde{c} - 1) \partial_{x_i} & \text{if } i \in \overline{1, m_1}, \\ x_0 \partial_{y_i} + x_i(\tilde{D} + \tilde{c}) & \text{if } i \in \overline{m_1 + 1, m_2}, \\ -x_0 y_i + x_i(\tilde{D} + \tilde{c}) & \text{if } i \in \overline{m_2 + 1, m}, \end{cases} \quad (3.61)$$

$$\pi_{c,S}(E_{m+1,m+1} - E_{2m+2,2m+2}) = \tilde{D} + x_0 \partial_{x_0} + \tilde{c}, \quad (3.62)$$

for  $i, j \in \overline{1, m}$ .

Recall  $\mathcal{B} = \mathbb{F}[x_1, \dots, x_n, y_1, \dots, y_n]$ . Set

$$\mathcal{B}_{\langle k \rangle} = \mathcal{A}_{\langle k \rangle} \bigcap \mathcal{B} \quad \text{for } k \in \mathbb{Z} \quad (3.63)$$

(cf. (2.6)). Then  $\mathcal{B} = \bigoplus_{k \in \mathbb{Z}} \mathcal{B}_{\langle k \rangle}$  is a  $\mathbb{Z}$ -graded space. The following result is due to [10]:

**Lemma 3.4.** *Assume  $m \geq 2$ . Let  $k \in \mathbb{Z}$ . If  $m_1 < m_2$  or  $k \neq 0$ , the subspace  $\mathcal{B}_{\langle k \rangle}$  is an irreducible  $\mathcal{K}$ -submodule (cf. (3.10)). When  $m_1 = m_2$ , the subspace  $\mathcal{B}_{\langle 0 \rangle}$  is a direct sum of two irreducible  $\mathcal{K}$ -submodules.*

In fact, any pair of the irreducible submodules in the above are not isomorphic  $\mathcal{K}$ -modules because they have distinct weight sets of singular vectors with respect to the Lie subalgebra  $\sum_{i,j=1}^m \mathbb{F}(E_{i,j} - E_{m+1+j,m+1+i}) \cong gl(m, \mathbb{F})$  (cf. [9]). When  $m = m_1 = m_2 = 1$ ,  $\mathcal{B} = \mathbb{F}[x_1, x_2]$  and

$$\pi_{c,S}(\mathcal{K}) = \mathbb{F}(x_1 \partial_{x_1} + y_1 \partial_{y_1} + 1) + \mathbb{F}x_1 y_1 + \mathbb{F} \partial_{x_1} \partial_{y_1}. \quad (3.64)$$

So all  $\mathcal{B}_{\langle k \rangle}$  with  $k \in \mathbb{Z}$  are irreducible  $\mathcal{K}$ -submodules. Recall the representation  $\pi_{c,S}^{\vec{0}}$  of  $sp(2m+2, \mathbb{F})$  is the representation  $\pi_{c,S}$  (cf. (3.52)-(3.62)) on  $\mathcal{A}$ . The following is the third result in this section.

**Theorem 3.5.** *The representation  $\pi_{c,S}^{\vec{0}}$  of  $sp(2m+2, \mathbb{F})$  is irreducible if  $c \notin \mathbb{Z}$ .*

*Proof.* Let  $\mathcal{M}$  be any nonzero  $sp(2m+2, \mathbb{F})$ -submodule of  $\mathcal{A}$ . Repeatedly applying  $E_{2m+2,m+1}$  to  $\mathcal{M}$  by the first equation in (3.57), we get

$$\mathcal{A} \cap \mathcal{B} \neq \{0\}. \quad (3.65)$$

According to (3.62),

$$\mathcal{B}_{\langle k \rangle} = \{f \in \mathcal{B} \mid (E_{m+1,m+1} - E_{2m+2,2m+2})(f) = (k + \tilde{c})f\}. \quad (3.66)$$

Thus

$$\mathcal{M} = \bigoplus_{k \in \mathbb{Z}} \mathcal{M} \cap \mathcal{B}_{\langle k \rangle}. \quad (3.67)$$

If  $\mathcal{M} \cap \mathcal{B}_{\langle 0 \rangle} \neq \{0\}$ , then (3.59) gives

$$(E_{2m+2,1} + E_{m+2,m+1})(\mathcal{M} \cap \mathcal{B}_{\langle 0 \rangle}) = x_1(\mathcal{M} \cap \mathcal{B}_{\langle 0 \rangle}) \subset \mathcal{M} \cap \mathcal{B}_{\langle -1 \rangle}. \quad (3.68)$$

Thus we always have  $\mathcal{M} \cap \mathcal{B}_{\langle k \rangle} \neq \{0\}$  for some  $0 \neq k \in \mathbb{Z}$ . According to Lemma 3.4 and (3.64),  $\mathcal{B}_{\langle k \rangle}$  is an irreducible  $\mathcal{K}$ -module. So

$$\mathcal{B}_{\langle k \rangle} \subset \mathcal{M}. \quad (3.69)$$

Next (3.58) yields

$$\mathcal{B}_{\langle k-r \rangle} = (\partial_{y_1})^r(\mathcal{B}_{\langle k \rangle}) = (E_{1,m+1} - E_{2m+2,m+2})^r(\mathcal{B}_{\langle k \rangle}) \subset \mathcal{M} \quad \text{for } r \in \mathbb{N}. \quad (3.70)$$

On the other hand, if  $\mathcal{B}_{\langle \ell \rangle} \subset \mathcal{M}$ , then the assumption  $c \notin \mathbb{Z}$  and the second equation in (3.57) give

$$x_0^r \mathcal{B}_{\langle \ell \rangle} = (E_{2m+2,m+1})^r(\mathcal{B}_{\langle \ell \rangle}) \subset \mathcal{M} \quad \text{for } r \in \mathbb{N}. \quad (3.71)$$

Suppose that for some  $s \in \mathbb{Z}$ ,

$$x_0^r \mathcal{B}_{\langle s \rangle}, x_0^r \mathcal{B}_{\langle s-1 \rangle} \subset \mathcal{M} \quad \text{for } r \in \mathbb{N}. \quad (3.72)$$

For any  $\ell \in \mathbb{N}$ ,

$$x_0^\ell \mathcal{B}_{\langle s+1 \rangle} = (\tilde{D} + \tilde{c} - 1) \partial_{x_1} (x_0^\ell \mathcal{B}_{\langle s \rangle}) = [E_{m+1,m+2} + E_{1,2m+2} - x_0 \partial_{y_1}] (x_0^\ell \mathcal{B}_{\langle s \rangle}) \quad (3.73)$$

by (3.61). Note

$$x_0 \partial_{y_1} (x_0^\ell \mathcal{B}_{\langle s \rangle}) = x_0^{\ell+1} \mathcal{B}_{\langle s-1 \rangle} \subset \mathcal{M}. \quad (3.74)$$

Thus (3.73) leads to

$$x_0^\ell \mathcal{B}_{(s+1)} \subset \mathcal{M}. \quad (3.75)$$

By (3.70)-(3.75) and induction on  $s$ , we prove

$$x_0^r \mathcal{B}_{(k)} \subset W \quad \text{for } x_0 \in \mathbb{N}, k \in \mathbb{Z}. \quad (3.76)$$

So  $\mathcal{M} = \mathcal{A}$ . Therefore,  $\mathcal{A}$  is an irreducible  $sp(2m+2, \mathbb{F})$ -module.  $\square$

**Remark 3.6.** The above irreducible representation depends on the three parameters  $c \in \mathbb{F}$  and  $m_1, m_2 \in \overline{1, n}$ . It is not highest-weight type because of the mixture of multiplication operators and differential operators in (3.55), (3.56) and (3.58)-(3.61). Since  $\mathcal{B}$  is not completely reducible as a module of the Lie subalgebra  $\sum_{i,j=1}^m \mathbb{F}(E_{i,j} - E_{m+1+j, m+1+i})$  by [9] when  $m \geq 2$  and  $m_1 < m$ ,  $\mathcal{A}$  is not a unitary  $sp(2m+2, \mathbb{F})$ -module. Expression (3.62) shows that  $\mathcal{A}$  is a weight  $sp(2m+2, \mathbb{F})$ -module with finite-dimensional weight subspaces.

*Case 4.*  $S \neq \emptyset$ ,  $\vec{a} \neq 0$ ,  $a_{i_0} \neq 0$  for some  $m+1+i_0 \in S \cap \overline{m+2, 2m+1}$  if  $S \cap \overline{m+2, 2m+1} \neq \emptyset$ , and  $a_{m+1+j_0} \neq 0$  for some  $j_0 \in S \cap \overline{1, m+1}$  if  $S \cap \overline{1, m+1} \neq \emptyset$ .

We take (3.49)-(3.62). By the above assumption,  $b_{j_0} \neq 0$  for some  $j_0 \in \overline{1, m_1}$ , and  $a_{i_0} \neq 0$  for some  $i_0 \in \overline{m_2+1, m}$  if  $m_2 < m$ . Recall the representation  $\pi_{c,S}^{\vec{a}}$  of  $sp(2m+2, \mathbb{F})$  is the representation  $\pi_{c,S}$  (cf. (3.52)-(3.62)) on  $\mathcal{A}_{\vec{a}}$  (cf. (1.7)). Under the assumption, we have the following fourth result in this section:

**Theorem 3.7.** *The representation  $\pi_{c,S}^{\vec{a}}$  of  $sp(2m+2, \mathbb{F})$  is irreducible for any  $c \in \mathbb{F}$ .*

*Proof.* Let  $\mathcal{M}$  be a nonzero  $sp(2m+2, \mathbb{F})$ -submodule of  $\mathcal{A}_{\vec{a}}$ . Take any  $0 \neq f e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$  with  $f \in \mathcal{A}$ . By the assumption,  $a_0 = 0$ . Repeatedly applying the first equation in (3.57) to  $f e^{\vec{a} \cdot \vec{x}}$  if necessary, we may assume  $f \in \mathcal{B} = \mathbb{F}[x_1, \dots, x_m, y_1, \dots, y_m]$ . Then (3.59) yields

$$(E_{2m+2,i} + E_{m+1+i, m+1} + a_i)(f e^{\vec{a} \cdot \vec{x}}) = -\partial_{x_i}(f) e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{m_1+1, m}. \quad (3.77)$$

Moreover, (3.58) yields

$$(E_{i, m+1} - E_{2m+2, m+1+i} - b_j)(f e^{\vec{a} \cdot \vec{x}}) = \partial_{y_j}(f) e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } j \in \overline{1, m_2}. \quad (3.78)$$

Repeatedly applying (3.77) and (3.78) if necessary, we can assume

$$f \in \mathbb{F}[x_1, \dots, x_{m_1}, y_{m_2+1}, \dots, y_m]. \quad (3.79)$$

According to (3.55),

$$(E_{i, m+1+j_0} + E_{j_0, m+1+i} - a_{j_0} b_i - a_i b_{j_0})(f e^{\vec{a} \cdot \vec{x}}) = (b_{j_0} \partial_{x_i} + b_i \partial_{x_{j_0}})(f) e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.80)$$

for  $i \in \overline{1, m_1}$ . Taking  $i = j_0$  in (3.80), we get  $\partial_{x_{j_0}}(f) e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$ . Substituting it to (2.80) for general  $i$ , we obtain

$$\partial_{x_i}(f) e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{1, m_1}. \quad (3.81)$$

Moreover, (3.56) yields

$$(E_{m+1+j, i_0} + E_{m+1+i_0, j} - a_j b_{i_0} - a_{i_0} b_j)(f e^{\vec{a} \cdot \vec{x}}) = (a_{i_0} \partial_{y_j} + a_j \partial_{y_{i_0}})(f) e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.82)$$

for  $j \in \overline{m_2+1, m}$ . Letting  $j = i_0$  in (3.84), we find  $\partial_{y_{i_0}}(f) e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$ . Substituting it to (3.82) for general  $j$ , we get

$$\partial_{y_j}(f) e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } j \in \overline{m_2+1, m}. \quad (3.83)$$

Repeatedly applying (3.81) and (3.83) if necessary, we obtain  $e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$ . Equivalently,  $\mathcal{A}_{\vec{a},0} \subset \mathcal{M}$  (cf. (2.13)).

Suppose that for some  $\ell \in \mathbb{N}$ ,  $\mathcal{A}_{\vec{a},k} \subset \mathcal{M}$  whenever  $\ell \geq k \in \mathbb{N}$ . For any  $ge^{\vec{a} \cdot \vec{x}} \in \mathcal{A}_{\vec{a},\ell}$ , (3.59) implies

$$(E_{2m+2,i} + E_{m+1+i,m+1} - y_i \partial_{x_0})(ge^{\vec{a} \cdot \vec{x}}) = x_i ge^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{1, m_1} \quad (3.84)$$

and (3.58) leads to

$$(E_{2m+2,m+1+i} - E_{i,m+1} + x_i \partial_{x_0})(ge^{\vec{a} \cdot \vec{x}}) = y_j ge^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } j \in \overline{m_2 + 1, m}. \quad (3.85)$$

Moreover, (3.55) gives

$$(E_{i,m+1+j_0} + E_{j_0,m+1+i})(ge^{\vec{a} \cdot \vec{x}}) = [b_{j_0} x_i + x_i \partial_{y_{j_0}} + (\partial_{x_{j_0}} + a_{j_0})(\partial_{y_i} + b_i)](g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.86)$$

if  $i \in \overline{m_1 + 1, m_2}$ , and

$$(E_{i,m+1+j_0} + E_{j_0,m+1+i})(ge^{\vec{a} \cdot \vec{x}}) = [b_{j_0} x_i - a_{j_0} y_i + x_i \partial_{y_{j_0}} - y_i \partial_{x_{j_0}}](g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.87)$$

if  $i \in \overline{m_2 + 1, m}$ . Note that the inductual assumption imply

$$[x_i \partial_{y_{j_0}} + (\partial_{x_{j_0}} + a_{j_0})(\partial_{y_i} + b_i)](g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.88)$$

if  $i \in \overline{m_1 + 1, m_2}$ , and

$$[-a_{j_0} y_i + x_i \partial_{y_{j_0}} - y_i \partial_{x_{j_0}}](g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.89)$$

by (3.87) if  $i \in \overline{m_2 + 1, m}$ . Thus

$$x_i ge^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } i \in \overline{m_1 + 1, m}. \quad (3.90)$$

On the other hand, (3.56) yields

$$(E_{m+1+j,i_0} + E_{m+1+i_0,j})(ge^{\vec{a} \cdot \vec{x}}) = (a_{i_0} y_j - b_{i_0} x_j + y_j \partial_{x_{i_0}} - x_j \partial_{y_{i_0}})(g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.91)$$

if  $j \in \overline{1, m_1}$ , and

$$(E_{m+1+j,i_0} + E_{m+1+i_0,j})(ge^{\vec{a} \cdot \vec{x}}) = [a_{i_0} y_j + y_j \partial_{x_{i_0}} + (\partial_{x_j} + a_j)(\partial_{y_{i_0}} + b_{i_0})](g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.92)$$

if  $j \in \overline{m_1 + 1, m_2}$ . Observe that the inductual assumption imply

$$(-b_{i_0} x_j + y_j \partial_{x_{i_0}} - x_j \partial_{y_{i_0}})(g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.93)$$

by (3.84) if  $j \in \overline{1, m_1}$ , and

$$[y_j \partial_{x_{i_0}} + (\partial_{x_j} + a_j)(\partial_{y_{i_0}} + b_{i_0})](g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad (3.94)$$

if  $j \in \overline{m_1 + 1, m_2}$ . Hence

$$y_j ge^{\vec{a} \cdot \vec{x}} \in \mathcal{M} \quad \text{for } j \in \overline{1, m_2}. \quad (3.95)$$

Moreover, (3.61) yields

$$\begin{aligned} & (E_{m+1,m+1+j_0} + E_{j_0,2m+2})(ge^{\vec{a} \cdot \vec{x}}) \\ &= [b_{j_0} x_0 + x_0 \partial_{y_{j_0}} + (\tilde{D} - \sum_{i=1}^{m_1} a_i x_i + \sum_{j=m_1+1}^m a_j x_j + \sum_{r=1}^{m_2} b_r y_r \\ & \quad - \sum_{s=m_2+1}^m b_s y_s + \tilde{c} + 1)(a_{j_0} + \partial_{x_{j_0}})](g)e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}. \end{aligned} \quad (3.96)$$

Note that

$$x_0 \partial_{y_{j_0}}(g) e^{\vec{a} \cdot \vec{x}} \in \mathcal{A}_{\vec{a}, \ell} \subset \mathcal{M}; \quad (\tilde{D} + \tilde{c} + 1)(\partial_{x_{j_0}}(g)) e^{\vec{a} \cdot \vec{x}} \in \mathcal{A}_{\vec{a}, \ell-1} \subset \mathcal{M}. \quad (3.97)$$

Now (3.84), (3.85), (3.90) and (3.95)-(3.97) imply  $x_0 g e^{\vec{a} \cdot \vec{x}} \in \mathcal{M}$ . Therefore,  $\mathcal{A}_{\vec{a}, \ell+1} \subset \mathcal{M}$ . By induction,  $\mathcal{A}_{\vec{a}, \ell} \subset \mathcal{M}$  for any  $\ell \in \mathbb{N}$ . So  $\mathcal{A}_{\vec{a}} = \mathcal{M}$ . Hence  $\mathcal{A}_{\vec{a}}$  is an irreducible  $sp(2m+2, \mathbb{F})$ -module.  $\square$

With respect to the restricted representation  $\pi_{c,S}^{\vec{0}}$ ,  $\mathcal{A}$  is an infinite-dimensional weight  $sp(2m+2, \mathbb{F})$ -module with finite-dimensional weight subspaces by (3.52)-(3.54) and (3.62). Now Theorem 2 follows from Theorems 3.2, 3.3, 3.5 and 3.7.

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